JOURNAL OF COMBINATORIAL THEORY (A) 19, 113–114 (1975)

Note

On Sets of Pairwise Disjoint Blocks in Steiner Triple Systems

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Communicated by Marshall Hall, Jr.

Received October 21, 1974

In a Steiner triple system with 19 points, each disjoint pair of blocks is contained in at least 43 quadruples of pairwise disjoint blocks. In a Steiner triple system with 25 points, each disjoint pair of blocks is contained in a pairwise disjoint quintuple of blocks. Theorems used are those of Connor on determinants based on intersecting and nonintersecting blocks of a BIBD, and of Turán on extremal graphs without triangles.

A Steiner triple system with 19 points is a balanced incomplete block design (BIBD) with parameters $(v, b, r, k, \lambda) = (19, 57, 9, 3, 1)$. Connor [1] showed that in BIDB, each $t \times t$ matrix defined for t blocks with diagonal entries $(r-k)(r-\lambda)$ and off-diagonal entries $\lambda k - rc_{ij}$, where blocks i and j have c_{ij} points in common, must have a nonnegative determinant. A paraphrase of Connor's theorem is that the quadratic form belonging to each such symmetric matrix must be positive semidefinite. Let X and Y be two disjoint blocks of a Steiner triple system as cited. There are exactly 9 blocks intersecting both X and Y. (Excluding X and Y), there are $6(9-3-1) = 30$ blocks intersecting X or Y but not both. This leaves $57 - 9 - 30 - 2 = 16$ blocks disjoint from both X and Y. The Connor matrices for this BIBD have entries 48 on the diagonal, and entries 3 and -6 off-diagonal for disjoint and intersecting pairs of blocks respectively. Let A be the Connor matrix for X, Y, and the 16 disjoint blocks. Partition rows and columns of A as $2 + 16$. Then A_{11} has off-diagonal 3's; A_{12} and its transpose A_{21} contain all 3's. Partially reduce the quadratic form of A by adding $-1/17$ times the upper two rows of A to the other 16 (and likewise on columns); this addition replaces A_{12} and A_{21} by zero matrices. In 16×16 A_{22}, $6/17$ is subtracted from entries by this reduction; 48, 3, and -6 all decreased by $6/17$ leave numbers proportional to 90, 5, and -12. In order that the quadratic form belonging to this transform of A_{22} be positive semidefinite, the sum of entries must be nonnegative. Solving

$$16 \cdot 90 + x \cdot 10 + (120 - x)(-24) = 0,$$

gives a root between 42 and 43. Thus we conclude that an arbitrary pair of disjoint blocks is contained in at least 43 pairwise disjoint quadruples of blocks.

Carrying out the same steps with $(v, b, r, k, \lambda) = (25, 100, 12, 3, 1)$, we find that 41 blocks are disjoint from X and Y, and among these there must be at least 459 disjoint pairs. Define the graph of 41 nodes with adjacency for disjoint blocks. Turán ([2], or [3, p. 17]) showed that a graph with n nodes and no triangle has at most $[n^2/4]$ edges. $459 > [41^2/4]$, hence blocks X and Y are in a pairwise disjoint quintuple of blocks.

These results can be generalized.

References

1. W. S. Connor, On the structure of balanced incomplete block designs, *Ann. Math. Statist.* **23** (1952), 57–71.
2. P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapok* **48** (1941), 436–452.
3. Frank Harary, "Graph Theory," Addison–Wesley, New York, 1969.